Testing isomorphism of circular-arc graphs in polynomial time

Roman Nedela

(joint work with Ilia Ponomarenko and Peter Zeman)

A graph is said to be *circular-arc* if the vertices can be associated with arcs of a circle so that two vertices are adjacent if and only if the corresponding arcs overlap. It is proved that the isomorphism of circular-arc graphs can be tested by the Weisfeiler-Leman algorithm after individualization of two vertices. The circular-arc graphs arise in many applications, see [6], and form a natural border between isomorphism complete classes of graphs and the classes admitting polynomial-time isomorphism testing [1]. An efficient algorithm recognizing circular-arc graphs was constructed in [6]. There are also polynomial-time algorithms testing isomorphism in various subclasses of circular-arc graphs, e.g., interval graphs, co-bipartite circular-arc graphs, proper circular-arc graphs, and Helly circular-arc graphs, see [3, 5]. However, the problem of testing isomorphism for the whole class remained open until the present time (for details, we refer the reader to a discussion in [3]). We proved the following theorem

Theorem 1. *Given an n-vertex circular-arc graph X and any graph Y, one can test whether X is isomorphic to Y in time polynomial in n.*

Our approach is based on the theory of *coherent configurations*, see [2]. Such a configuration can be viewed as an arc-colored complete directed graph with a regularity condition. Namely, the number of triangles with fixed base and fixed colors of the sides depends on the color of the base (and does not depend on the chosen base). These numbers form the intersection number array of the coherent configuration. To every graph X, one can associate a unique coherent configuration $WL(X)$ such that the edge set of X is a union of color classes of $WL(X)$. The crucial part in the proof of Theorem 1 is that if X is a circular-arc graph with two distinguished non-adjacent vertices, then the intersection number array of $WL(X)$ form a full set of invariants of X with respect to isomorphism. Thus to test whether the graph X is isomorphic to a graph Y, it suffices to verify whether there exist two non-adjacent vertices of Y such that the intersection arrays of the corresponding coherent configurations are equal. Given an n-vertex graph X, the coherent configuration $WL(X)$ (and the corresponding intersection number array) can be constructed by the Weisfeiler-Leman algorithm in time $O(n^3 \log n)$ [4]. Since there are at most $\binom{n}{2}$ pairs of non-adjacent vertices in X, the running time of our algorithm is $O(n^5 \log n)$. Although the algorithm does not construct an isomorphism of two isomorphic graphs, it seems that the induction used in the proof of Theorem 1 can be modified to find the isomorphism (and even the set of all of them) explicitly.

References

[1] S. Chaplick, P. Zeman, GI-completeness of H-graphs (2019), in preparation.

[2] G. Chen, I. Ponomarenko, Lectures on coherent configurations (2019) `http://www.pdmi.ras.ru/~inp/ccNOTES.pdf`.

[3] A.R. Curtis, M.C. Lin, R.M. McConnell, Y. Nussbaum, F.J. Soulignac, J.P. Spinrad, J.I. Szwarcfiter, Isomorphism of graph classes related to the circular-ones property, Discrete Math. Theor. Comput. Sci. 15 (2013) 157–182.

[4] N. Immerman, E. Lander, Describing graphs: a first-order approach to graph canonization, Complexity Theory Retrospective, Springer-Verlag (1990) 59–81.

[5] J. Köbler, S. Kuhnert, O. Verbitsky, On the isomorphism problem for Helly circular-arc graphs, Inform. and Comput. 247 (2016) 266–277.

[6] A. Tucker, An efficient test for circular-arc graphs, SIAM J. Comput. 9 (1980) 1–24.

[7] B. Weisfeiler (editor), On construction and identification of graphs, Lecture Notes in Math. 558 (1976).