Tough enough H-graphs are Hamiltonian

Adam Kabela

(joint work with Tomáš Kaiser)

The present study is motivated by the conjecture of Chvátal [6] suggesting that there is a constant t such that every t-tough graph (on at least 3 vertices) is Hamiltonian. For an introduction to the topic, we refer the reader to [2, 3, 12]. We say that a graph is *t-tough* if for every integer c greater than 1, it cannot be disconnected into at least c components by removing less than $\lceil ct \rceil$ vertices. Given a graph G, we say that a graph S is its *underlying graph* if there is a family of connected subgraphs of S such that G is isomorphic to the intersection graph of the family. We recall that the conjecture was shown to be true when restricted, for instance, to interval graphs [4, 10], circular arc graphs [13], split graphs [11], spider graphs [9] or chordal graphs [5, 8]; and all these classes admit a natural definition in terms of underlying graphs. We show that for every k there is t such that being t-tough implies Hamiltonicity for every graph (on at least 3 vertices) whose some underlying graph has at most k cycles. The idea of the proof is to balance between two approaches: a direct use of the classical theorem of Chvátal and Erdős [7], and a generalized version of the technique of [8] (using the hypergraph extension of Hall's theorem by Aharoni and Haxell [1]).

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