On the achromatic number of the Cartesian product of two complete graphs

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Let G be a graph and C a finite set of colours. A proper vertex colouring $f : V(G) \to C$ is *complete* provided that for any pair c_1, c_2 of distinct colours in C there is a pair v_1, v_2 of adjacent vertices in G such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number of* G is the maximum cardinality of a set C admitting a complete proper vertex colouring of G, in symbols $\mathrm{achr}(G)$. Given two graphs G and H, we use $G \square H$ to denote the Cartesian product of G and H.

Theorem 1. *Let r be a projective plane order, and let p, q, k be integers satisfying $p = r^2 + r + 1$, $q \geq (r^3 + 1)(r + 1)$, $0 \leq k \leq r$ and $q \equiv k \pmod{r+1}$.*

1. *If $k = 0$, then* $\mathrm{achr}(K_p \square K_q) = \frac{pq}{r+1}$.
2. *If $k \geq 1$, then* $p\lfloor \frac{q}{r+1} \rfloor + k \leq \mathrm{achr}(K_p \square K_q) \leq p\lfloor \frac{q}{r+1} \rfloor + kr$.